1.

3. Danu sumu možemo prikazati na ovaj način:

$$\begin{aligned}
&\frac{2}{2\cdot 5}+\frac{2}{5\cdot 8}+\ldots+\frac{2}{1997\cdot 2000}+\frac{2}{2000\cdot 2003}\\
=\;&\frac{2}{3}\left(\frac{5-2}{2\cdot 5}+\frac{8-5}{5\cdot 8}+\ldots+\frac{2000-1997}{1997\cdot 2000}+\frac{2003-2000}{2000\cdot 2003}\right)\\
=\;&\frac{2}{3}\left(\frac{1}{2}-\frac{1}{5}+\frac{1}{5}-\frac{1}{8}+\ldots+\frac{1}{1997}-\frac{1}{2000}+\frac{1}{2000}-\frac{1}{2003}\right)\\
=\;&\frac{2}{3}\cdot\left(\frac{1}{2}-\frac{1}{2003}\right)=\frac{667}{2003}\,.
\end{aligned}$$

25 bodova

2.

4. Jednakost

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=1$$

množimo redom s a, b, c i dobivene jednakosti zbrojimo. 10 bodova

Dobiva se

$$\begin{aligned}
&\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\\
=\;&a+b+c-\frac{ab+bc}{c+a}-\frac{ac+bc}{a+b}-\frac{ab+ac}{b+c}\\
=\;&0.
\end{aligned}$$

15 bodova

3.

1. Kako je

$$\begin{aligned}
f(x)&=\frac{1}{\sqrt[3]{x^2+2x+1}+\sqrt[3]{x^2-1}+\sqrt[3]{x^2-2x+1}}\\
&=\frac{1}{\sqrt[3]{(x+1)^2}+\sqrt[3]{(x+1)(x-1)}+\sqrt[3]{(x-1)^2}}\cdot\frac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{\sqrt[3]{x+1}-\sqrt[3]{x-1}}\\
&=\frac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{(\sqrt[3]{x+1})^3-(\sqrt[3]{x-1})^3}=\frac{1}{2}\left(\sqrt[3]{x+1}-\sqrt[3]{x-1}\right),
\end{aligned}$$

15 bodova

imamo

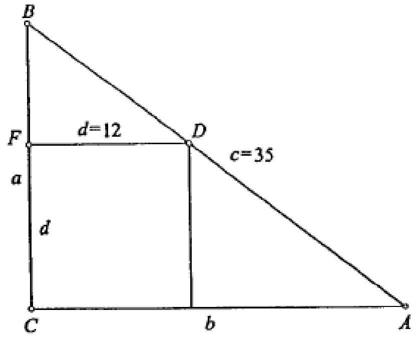
$$\begin{aligned}
&f(1)+f(2)+\ldots+f(2002)+f(2003)\\
=\;&\frac{1}{2}\left(\sqrt[3]{2}-\sqrt[3]{0}+\sqrt[3]{3}-\sqrt[3]{1}+\ldots+\sqrt[3]{2003}-\sqrt[3]{2001}+\sqrt[3]{2004}-\sqrt[3]{2002}\right)\\
=\;&\frac{1}{2}\left(\sqrt[3]{2004}+\sqrt[3]{2003}-1\right).
\end{aligned}$$

10 bodova

4.

4. Ako su a i b katete i $c = 35$ hipotenuza, vrijedi

$$a^2 + b^2 = 35^2 = 1225 \qquad \text{5 bodova}$$



S druge strane, iz sličnosti trokuta ABC i DBF dobivamo

$$\frac{a}{b} = \frac{a-d}{d} \implies d(a+b) = ab \quad \text{tj.} \quad 12(a+b) = ab. \qquad \text{5 bodova}$$

Riješimo sustav $a^2 + b^2 = 1225, \quad 12(a+b) = ab$. Kako je $a^2 + b^2 = (a+b)^2 - 2ab$, stavimo $x = a + b$, $y = ab$. Tada je

$$x^2 - 2y = 1225, \quad 12x = y \implies x^2 - 24x - 1225 = 0 .$$

Rješenja ove kvadratne jednadžbe su

$$x_1 = 49, \quad x_2 = -25 , \quad \text{i dalje,} \quad y_1 = 588 , \quad y_2 = -300 .$$

Zadovoljava samo $x_1 = 49$ i $y_1 = 588$. 5 bodova

Sada je $a + b = 49$ i $ab = 588$, pa su a i b rješenja kvadratne jednadžbe

$$t^2 - 49t + 588 = 0 .$$

Slijedi $a_1 = 21$, $b_1 = 28$ ili $a_2 = 28$, $b_2 = 21$, pa su duljine kateta danog trokuta 21 i 28. 10 bodova

5.

3. Kako 0 nije rješenje jednadžbe, dijeljenjem s x^2 dobije se

$$x^2 - x - 10 + \frac{2}{x} + \frac{4}{x^2} = 0 . \qquad \text{5 bodova}$$

Uvedemo li supstituciju $t = x - \frac{2}{x}$, tada je $t^2 = x^2 - 4 + \frac{4}{x^2}$, tj. $x^2 + \frac{4}{x^2} = t^2 + 4$.

Jednadžba sada poprima oblik

$$t^2 + 4 - t - 10 = 0 \quad \text{tj.} \quad t^2 - t - 6 = 0. \qquad \text{5 bodova}$$

Rješenja ove jednadžbe su $t_1 = 3$, $t_2 = -2$. 5 bodova

Za $t = 3$ je $x - \frac{2}{x} = 3$, tj. $x^2 - 3x - 2 = 0$ i rješenja su $x_{1,2} = \frac{3 \pm \sqrt{17}}{2}$. 5 bodova

Za $t = -2$ je $x - \frac{2}{x} = -2$, tj. $x^2 + 2x - 2 = 0$ i rješenja su $x_{3,4} = -1 \pm \sqrt{3}$. 5 bodova

6

1. *Prvo rješenje.* Koristeći svojstvo funkcije,

$$f(x+1) = \frac{1+f(x)}{1-f(x)}, \qquad \text{za } x \in \mathbf{Z}, \tag{1}$$

i početni uvjet

$$f(1) = 2, \tag{2}$$

dobivamo

$$\begin{aligned}
f(2) &= \frac{1+f(1)}{1-f(1)} = -3\,,\\
f(3) &= \frac{1+f(2)}{1-f(2)} = -\frac{1}{2}\,,\\
f(4) &= \frac{1+f(3)}{1-f(3)} = \frac{1}{3}\,,\\
f(5) &= \frac{1+f(4)}{1-f(4)} = 2 = f(1)\,,\\
f(6) &= \frac{1+f(5)}{1-f(5)} = \frac{1+f(1)}{1-f(1)} = -3 = f(2)\,,\\
f(7) &= \frac{1+f(6)}{1-f(6)} = \frac{1+f(2)}{1-f(2)} = -\frac{1}{2} = f(3)\,,\\
f(8) &= \frac{1+f(7)}{1-f(7)} = \frac{1+f(3)}{1-f(3)} = \frac{1}{3} = f(4)\,,
\end{aligned}$$

Odavde slijedi $f(n+4) = f(n)$ za $n \in \mathbf{N}$. 15 bodova

Konačno je

$$f(2004) = f(2000) = f(1996) = \ldots = f(4) = \frac{1}{3}\,.$$ 10 bodova

Drugo rješenje. Za prirodan broj n i danu funkciju f vrijedi:

$$\begin{aligned}
f(n+1) &= \frac{1+f(n)}{1-f(n)}\,,\\
f(n+2) &= \frac{1+f(n+1)}{1-f(n+1)} = \frac{1+\dfrac{1+f(n)}{1-f(n)}}{1-\dfrac{1+f(n)}{1-f(n)}} = -\frac{1}{f(n)}\,,\\
f(n+4) &= -\frac{1}{f(n+2)} = f(n)\,.
\end{aligned}$$

Dakle, funkcija f ima svojstvo $f(n+4) = f(n)$. 15 bodova

Sada je $f(2004) = f(2000) = f(1996) = \ldots = f(4) = \frac{1}{3}$. 10 bodova